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ON STABILITY IN ORDINARY DIFFERENTIAL EQUATIONS

by

Ronald Albert Rinaldi

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THESIS

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On Stability in Ordinary Differential Equations

by

Ronald Albert Rinaldi
Lieutenant (junior grade), United States Navy
B.S., United States Naval Academy, 1968

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requirements for the degree of

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ABSTRACT

Stability in the sense of Lyapunov is investigated for systems of ordinary differential equations. Various established results are discussed and a new set of sufficient conditions for a stability comparison theorem is obtained. The treatment is principally concerned with linear systems, though some results for non-linear systems are considered briefly.

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I. INTRODUCTION

Stability was originally a physical concept, believed first to have been considered by Lagrange in connection with the equilibrium states of conservative systems. Equilibrium states are those in which the velocities and accelerations are zero, describing a state of rest. A system in equilibrium may react in one of two manners when influenced externally; it may exhibit motions which remain in a small region about the equilibrium position (which does not preclude the possibility that the system return to its state of equilibrium), or it may exhibit motions which cause the system to increasingly deviate from its equilibrium state, never returning to it. In the former case, the equilibrium state is said to be stable, while the latter case describes an unstable state of equilibrium.

The above remarks are, of course, not rigorous and are intended solely as an intuitive guide to the meaning of stability of equilibrium states of physical systems. Physical systems are almost always described by a system of differential equations and in studying the behavior of a physical system, we ultimately study the behavior of the solutions to the differential equations which describe the system. Thus the study of stability of a physical system is reduced to a consideration of the stability of solutions of differential equations, and stability theory is today an integral part of that segment of analysis known as the theory of ordinary differential equations.

Stability theory is quite advanced today and there are numerous concepts and approaches which vary with both the form and order of the system of differential equations. For example, the Poincare-Bendixson theory is an analytical-topological approach which is particularly useful for second order autonomous systems, while for non-autonomous systems, the concept of stability in the sense of Lyapunov is utilized and the approach is purely analytical. This paper is concerned with a general study of stability in the sense of Lyapunov, with particular emphasis on systems of linear differential equations though some results for nonlinear systems are considered. Section II deals with a review of familiar facts from the theory of linear ordinary differential equations. Stability and boundedness for linear systems are discussed in section III. Non-homogeneous systems are considered in section IV, and the concepts of uniform and restrictive stability are treated in section V. Some results for non-linear systems are given in section VI, and concluding remarks follow in section VII.

II. PRELIMINARY REMARKS

We consider the behavior of solutions of the nth order linear non-homogeneous system

(2.1) $$x' = A(t)x$$

where $x = x(t)$ is an n by 1 column vector and $A(t)$ is an n by n matrix with elements $a_{ij}(t)$, $i,j = 1,\ldots n$, which are real valued functions continuous on the half-line $[t_0,\infty)$.

The norm of x, A is defined by

(2.2i) $$||x(t)|| = \Sigma|x_i(t)|$$

(2.2ii) $$||A(t)|| = \Sigma|A_{ij}(t)|$$

and note that both $||x||$ and $||A||$ are functions of t. The usual properties of norms are satisfied and are stated here without proof. For $||x||$ they are,

(2.3i) $$||x|| \geq 0 \quad \text{and } ||x|| = 0 \quad \text{iff } x(t) \equiv 0,$$

(2.3ii) $$\Big|\, ||x|| - ||y|| \,\Big| \leq ||x \pm y|| \leq ||x|| + ||y||,$$

(2.3iii) $$||\alpha x|| = |\alpha|\ ||x|| \quad \text{for } \alpha \text{ a real number},$$

(2.3iv) $$||\textstyle\int x(t)dt|| \leq \int ||x(t)||dt,$$

and for $||A||$,

(2.4i) $$||A|| \geq 0 \quad \text{and } ||A|| = 0 \quad \text{iff } A(t) \equiv 0,$$

(2.4ii) $$\Big|\, ||A|| - ||B|| \,\Big| \leq ||A \pm B|| \leq ||A|| + ||B||,$$

(2.4iii) $$||AB|| \leq ||A||\ ||B||,$$

(2.4iv) $$||Ax|| \leq ||A||\ ||x||,$$

(2.4v) $$||\alpha A|| = |\alpha|\ ||A|| \quad \text{for } \alpha \text{ a real number}.$$

Capital letters will denote n by n matrices, lower case letters n by 1 column vectors, and Greek letters real numbers unless otherwise noted. Primes (') denote differentiation with respect to t; A^T denotes the transpose of A; tr(A) denotes the trace of A; det A denotes the determinant of A; and adj A decotes the adjoint of A. Lower subscripts denote components of a vector or matrix, for example, x_i is the ith component of the vector x.

The questions of existence and uniqueness of solutions to the system (2.1) may be dealt with quite briefly. Solutions to (2.1) satisfying the initial condition $x(t_o) = x_o$ exist and are unique on any interval in which A(t) is continuous. Hereafter, continuity of A(t) on $[t_o,\infty)$ will be assumed so that existence and uniqueness of solutions for systems of this form are always assured.

Linear homogeneous systems of differential equations are characterized by the fact that any linear combination of solutions is also a solution. The space of all possible solutions to the system (2.1) constitutes an n-dimensional vector space. Thus there exists n linearly independent solutions, $\phi^1, \phi^2, \ldots, \phi^n$, which form a basis for this vector space of solutions. Any particular solution x(t) of (2.1) can then be expressed as a linear combination of elements in this basis. Since any n linearly independent solutions constitute a basis, it is convenient to pick a particular set and define the fundamental matrix X(t) as follows.

(2.5) $$X(t) = \phi^i_j(t) \quad (i,j = 1,\ldots,n),$$

i.e., the columns of X(t) are n linearly independent solutions of (2.1) and these n solutions ϕ^i $(i = 1,\ldots,n)$ are chosen to satisfy the initial condition $\phi^i_j(t_o) = \delta_{ij}$ $(i,j = 1,\ldots,n)$ so that $X(t_o) = I$, the identity matrix.

From (2.5), we note that $X(t)$ satisfies the matrix differential equation

$$X' = A(t)X, \tag{2.6}$$

and that any solution $x(t;t_0,x_0)$ of (2.1) satisfying the initial condition $x(t_0) = x_0$ may be written

$$x(t) = X(t)x_0. \tag{2.7}$$

From the formula for the derivative of the determinant of a matrix,

$$\frac{d}{dt} \det X(t) = \operatorname{tr}\,[A(t)] \det X(t),$$

we obtain the Jacobi-Liouville formula

$$\begin{aligned} \det X(t) &= \det X(t_0) \exp \int_{t_0}^{t} \operatorname{tr} A(s)\, ds \\ &= \exp \int_{t_0}^{t} \operatorname{tr} A(s)\, ds, \end{aligned} \tag{2.8}$$

and note from (2.8) that $X(t)$ is non-singular for $t \geq t_0$.

Any non-singular matrix $W(t)$ which is a solution of (2.6) may serve as a basis matrix though if $W(t_0) \neq I$, we no longer have (2.7) but instead

$$x(t;t_0,x_0) = W(t)\, w_0,$$

where $w_0 \neq x_0$.

From the formula for the derivative of the inverse of a matrix,

$$\frac{d}{dt} [X^{-1}(t)] = -X^{-1}(t)\, X'(t)\, X^{-1}(t),$$

we obtain the result that if $Y(t) = X^{-1}(t)^T$, then $Y(t)$ satisfies the matrix differential equation

$$Y' = -A(t)^T\, Y, \tag{2.9}$$

and since $Y(t)$ is non-singular and $Y(t_0) = I$, we note that $Y(t)$ is the fundamental matrix to the adjoint system of (2.1) given by

$$x' = -A(t)^T x. \tag{2.10}$$

The fundamental matrix $X(t)$ provides an explicit form for the solution of a non-homogeneous system of the form

$$x' = A(t)\, x + b(t) \tag{2.11}$$

as the following proposition illustrates.

<u>Proposition 2.1</u>. The solution $x(t;t_0,x_0)$ of the system (2.11), where $A(t)$ and $b(t)$ are continuous on $[t_0,\infty)$ may be written

$$x(t) = X(t)\, x_0 + \int_{t_0}^{t} X(t)\, X^{-1}(s)\, b(s)ds,$$

where $X(t)$ is the fundamental matrix of the homogeneous system $x' = A(t)\, x$.

Proof: Direct substitution.

A similar formula may be obtained for non-linear mixed systems of the form

$$x' = A(t)\, x + f(t,x). \tag{2.12}$$

<u>Proposition 2.2</u>. Let $f(t,x)$ be continuous for $t \geq t_0$ and $||x|| < \xi$, and $A(t)$ continuous for $t \geq t_0$. Then the solution $x(t;\, t_0,x_0)$ of (2.12) may be written

$$x(t) = X(t)\, x_0 + \int_{t_0}^{t} X(t)\, X^{-1}(s)\, f(s,x(s))\, ds,$$

where $X(t)$ is the fundamental matrix of $x' = A(t)\, x$.

Proof: Direct substitution.

Both Propositions 2.1 and 2.2 simplify in the case $A(t) = A$, a constant matrix and $t_0 = 0$. For then, the matrix $X(t)\, X^{-1}(s)$

satisfies (2.6) with initial condition

$$X(t)\, X^{-1}(s) = I$$

at $t = s$, and the same follows for $X(t-s)$. Hence by uniqueness,

$$X(t)\, X^{-1}(s) = X(t-s), \tag{2.13}$$

for A a constant matrix and $X(0) = I$.

Other well known results from the theory of ordinary differential equations will be stated when needed.

III. LINEAR HOMOGENEOUS SYSTEMS

Consider a general first order system of ordinary differential equations of the form

$$(3.1) \qquad x' = f(t,x)$$

where $x(t)$ is an n by 1 column vector and $f(t,x)$ is an n by 1 column vector defined and continuous in the region $\Gamma = \{(t,x): 0 \leq t < \infty,\ ||x|| < \infty\}$.

We wish to define a concept of stability for this system. One definition which immediately comes to mind might be to define the system to be stable if it exhibits only finite motions, i.e., all solutions remain bounded as $t \to \infty$. However, such a concept is not particularly useful as it distinguishes variational effects only as to their being finite of not. For example, an earthquake would be stable in this sense.

What we wish to embody in a stability concept is a means by which we can measure the effects of small variations on the system (or a particular solution of the system). One criterion we would desire of a stable system is that small variations not affect the system in a drastic manner. This is the primary consideration embodied in the concept of Lyapunov stability, which imposes stringent restrictions on the solutions of (3.1), by requiring that solutions which are initially close together remain close together for all future time. Formally, we define stability as follows.

Definition 3.1. Let $x(t;t_0,x_0)$ be a solution of (3.1) satisfying

(3.2i) $x(t)$ exists for $t \geq t_0$,

and

(3.2ii) $(t,x(t)) \in \Gamma$ for $t \geq t_0$.

Then $x(t)$ is said to be stable (in the sense of Lyapunov) if

(3.2iii) there exists $\alpha>0$ such that every solution $y(t;t_0,y_0)$ exists on $[t_0,\infty)$ and $(t,y(t))$ whenever $||x_0 - y_0|| < \alpha$,

and

(3.2iv) given $\varepsilon>0$ there exists $\delta = \delta(\varepsilon,f,x_0)$, $0<\delta<\alpha$, such that $||x(t) - y(t)|| < \varepsilon$ for $t \geq t_0$ for all solutions $y(t)$ with $||x_0 - y_0|| < \delta$.

Strictly speaking, the above defines stability at the right, since the behavior as $t\to\infty$ is considered; replacing the interval $[t_0,\infty)$ by the interval $(-\infty,t_0]$ would define stability at the left. Note from (3.2iv) that stability is equivalent to uniform continuity of $x(t;t_0,x_0)$ on $[t_0,\infty)$ with respect to the initial vector x_0. A solution which is not stable is said to be unstable. Hereafter, stability will always refer to Lyapunov stability at the right as defined above.

A special case of stability occurs when solutions which are initially close approach one another as $t\to\infty$. This leads to the following definition.

Definition 3.2. A solution $x(t;t_0,x_0)$ of (3.1) is said to be asymptotically stable on $[t_0,\infty)$ if it is stable and in addition there exists $\delta' > 0$ such that $||x(t) - y(t)||\to 0$ as $t\to\infty$ whenever $y(t;t_0,y_0)$ is a solution with $||x_0 - y_0|| < \delta'$.

Note that by definition, a solution which is asymptotically stable is stable. If all solutions of the system (3.1) are (asymptotically) stable, then we say that the system is (asymptotically) stable, while if there exists an unstable solution to (3.1), we will call the system unstable.

The following examples illustrate some of these concepts.

Example 3.1. Consider the one-dimensional system

$$x' = -x$$

on $[0,\infty)$. The general solution $x(t;0,x_0)$ is given by $x(t) = x_0 e^{-t}$, so that

$$||x(t) - y(t)|| = |x_0 - y_0| e^{-t} \leq |x_0 - y_0|$$

Conditions (3.2 i,ii, and iii) are satisfied as is (3.2iv) with $\delta = \varepsilon$. Thus the system is stable.

In addition,

$$||x(t) - y(t)|| = |x_0 - y_0| e^{-t} \to 0 \quad \text{as } t\to\infty$$

so that the system is asymptotically stable.

Example 3.2. Every non-trivial solution of the one-dimensional system

$$x' = 0 \quad \text{on } [0,\infty)$$

is of the form $x(t;0,x_0) = x_0$ and thus

$$||x(t) - y(t)|| = |x_0 - y_0|.$$

Conditions (3.2i,ii, and iii) are satisfied as is (3.2iv), again with $\delta = \varepsilon$.

However,

$$\lim_{t\to\infty} ||x(t) - y(t)|| = |x_0 - y_0| > 0,$$

so that no solution is asymptotically stable. Thus the system is stable.

Example 3.3. Non-trivial solutions of the one-dimensional system

$$x' = x \quad \text{on } [0,\infty)$$

are of the form $x(t;0,x_0) = x_0 e^t$ and thus

$$||x(t) - y(t)|| = |x_0 - y_0| e^t$$

which can be made arbitrarily large independent of $|x_0 - y_0|$ by taking t large enough. Thus the system is unstable.

Example 3.4. Cesari [2] shows that the trivial solution $z(t) \equiv 0$ of the one dimensional system

$$x' = x^2 \quad \text{on } [0,\infty)$$

is not stable since every (real) solution $x(t;0,x_0)$ is of the form

$$x(t) = \frac{x_0}{(1-tx_0)}$$

and solutions with $x_0 > 0$ cease to exist at $t = (x_0)^{-1}$ in violation of (3.2iii).

For linear systems of the form

$$x' = A(t)\, x + b(t) \tag{3.3}$$

with $A(t)$ and $b(t)$ continuous on $[t_0,\infty)$, (3.2i,ii, and iii) are always satisfied so that (3.2iv) becomes the determining factor as to whether or not solutions to these systems are stable. For these systems, we have the following useful proposition.

Proposition 3.1. If the linear system (3.3) has a stable solution then all its solutions are stable, i.e., the system is stable.

Proof: Let $x(t;t_0,x_0)$ be a stable solution and $y(t;t_0,y_0)$ be any other solution. Let $\varepsilon > 0$.

If $z(t;t_0,z_0)$ is a solution of (3.3), then $x(t) - y(t) + z(t)$ is also a solution since the system is linear. Then since $x(t)$ is stable, there exists $\delta > 0$ such that if

$$||x(t_0) - [x(t_0) - y(t_0) + z(t_0)]|| = ||y_0 - z_0|| < \delta$$

then

$$||x(t) - [x(t) - y(t) + z(t)]|| = ||y(t) - z(t)|| < \varepsilon$$

which says that y(t) is stable. Thus all solutions are stable.

Corollary 1. Proposition 3.1 remains true if stable is replaced by asymptotically stable.

Proof: The proof is identical to that above.

We next define the concept of boundedness of solutions to 3.1.

Definition 3.3. A solution x(t) of (3.1) is said to be bounded if there exists $M > 0$ such that $||x(t)|| \leq M$, $t \geq t_0$, i.e., if $||x(t)||$ is a bounded function of t on $[t_0,\infty)$. If all solutions to (3.1) are bounded, we say that the system is bounded.

Stability and boundedness are independent concepts as the following examples illustrate.

Example 3.5. All solutions of the one dimensional system

$$x' = 2t \quad \text{on } [0,\infty)$$

are of the form $x(t;t_0,x_0) = x_0 + t^2$ and are stable yet unbounded on $[0,\infty)$.

Example 3.6. Cesari [2] shows that every non-trivial solution of the two dimensional system

$$x' = y$$

$$y' = -1/2\, x\, (x^2 + (x^4 + 4y^2)^{1/2},$$

or equivalently,

$$z'' = -1/2\, z\, (z^2 + (z^4 + 4(z')^2)^{1/2}$$

is of the form

$$z(t) = c \sin (ct+d), \quad c \neq 0$$

and is thus bounded; none of these solutions is stable.

For linear homogeneous systems of the form

$$x' = A(t)\, x \tag{3.4}$$

stability and boundedness are equivalent as the following theorem illustrates.

Theorem 3.1. The system (3.4) is stable iff it is bounded.

Proof: Assume the system is stable. To show that it is bounded, it suffices to show that there exists $M > 0$ such that $||X(t)|| \leq M$ where $X(t)$ is the fundamental matrix of the system. Since the system is stable, in particular, the trivial solution $z(t) \equiv 0$ is stable. Thus if $\varepsilon > 0$ there exists $\delta > 0$ such that every solution $x(t;t_0,x_0)$ with $||x_0|| < \delta$ satisfies $||x(t)|| < \varepsilon$ for $t \geq t_0$. In particular, let $x(t)$ be that solution with initial vector x_0 given by

$$x_0 = \frac{\delta}{2} e^i$$

where e^i is the vector whose jth component is given by $e^i_j = \delta_{ij}$. Then $||x_0|| < \delta$ so that

$$||x(t)|| = ||X(t)\, x_0|| < \varepsilon, \quad t \geq t_0.$$

But for this choice of x_0, $X(t)\, x_0 = \frac{\delta}{2} \phi^i$, where ϕ^i is the ith column of $X(t)$. Thus

$$\frac{\delta}{2} ||\phi^i(t)|| < \varepsilon \quad t \geq t_0.$$

This procedure is valid for $i = 1,\ldots,n$ so that

$$||X(t)|| = \sum_{i=1}^{n} ||\phi^i(t)|| < \frac{2n\varepsilon}{\delta} = M$$

and hence $X(t)$ is bounded. Thus the system is bounded

Conversely, assume the system is bounded so that $X(t) \leq M$, for $t \geq t_0$. Then if $x(t;t_0,x_0)$ and $y(t;t_0,y_0)$ are two solutions to (3.4)

$$||x(t) - y(t)|| = ||X(t)(x_0 - y_0)|| \leq M\, ||x_0 - y_0||$$

so that if $||x_0 - y_0|| < \frac{\varepsilon}{M}$, then $||x(t) - y(t)|| < \varepsilon$ for $t \geq t_0$ which shows that the system is stable.

Corollary 1. The system $x' = A(t)\, x$ is asymptotically stable iff $X(t) \to 0$ as $t \to \infty$, where $X(t)$ is the fundamental matrix of the system.

Proof: The reverse implication is obvious. To show the forward implication, note that the hypotheses imply that the trivial solution is asymptotically stable and the result then follows as in the proof of the first half of Theorem 3.1.

Before proceeding to consider various results concerning stability in linear homogeneous systems, it is convenient to introduce the following lemma, known as Gronwall's Lemma, which is an extremely useful tool in stability theory.

Lemma 3.1. If $f(t)$ and $g(t)$ are real valued non-negative functions, $f(t)$ is continuous on $[t_0,\infty)$ and $g(t)$ is integrable in every finite interval contained in $[t_0,\infty)$ and if for some constant $\alpha \geq 0$

$$f(t) \leq \alpha + \int_{t_0}^{t} f(s)\, g(s)\, ds, \tag{3.5}$$

then

$$f(t) \leq \alpha \exp \int_{t_0}^{t} g(s)\, ds. \tag{3.6}$$

Proof: If $\alpha > 0$, then we have

$$\frac{f(t)\, g(t)}{\alpha + \int_{t_0}^{t} f(s)g(s)ds} \leq g(t).$$

Integrating the above between the limits t_0 and t yields

$$\ln [\alpha + \int_{t_0}^{t} f(s)g(s)ds] - \ln \alpha \leq \int_{t_0}^{t} g(s)ds,$$

or taking exponents,

$$\alpha + \int_{t_0}^{t} f(s)g(s)ds \leq \alpha \exp \int_{t_0}^{t} g(s)ds,$$

and thus

$$f(t) \leq \alpha \exp \int_{t_0}^{t} g(s)ds.$$

If $\alpha = 0$, then (3.5) holds for all real numbers $\beta > 0$, and thus (3.6) holds which implies as $\beta \to 0$ that $f(t) \equiv 0$. This completes the proof.

Utilizing Lemma 3.1, it is possible to prove the following proposition.

Proposition 3.2. If $\int_{t_0}^{\infty} ||A(t)||\, dt < \infty$, then the system $x' = A(t)\, x$ is stable.

Proof: Let X(t) be the fundamental matrix of the system. Then from (2.6)

$$X(t) = I + \int_{t_0}^{t} A(s)\, X(s)\, ds,$$

so that

$$||X(t)|| \leq n + \int_{t_0}^{t} ||A(s)||\, ||X(s)||\, ds.$$

Applying Gronwall's Lemma yields

$$||X(t)|| \leq n \exp \int_{t_0}^{t} ||A(s)||\, ds \leq n \exp \int_{t_0}^{\infty} ||A(s)||\, ds$$

and thus X(t) is bounded. By Theorem 3.1, the system is stable.

We next consider the linear homogeneous system with constant coefficients

(3.7) $$x' = A x$$

whose fundamental matrix $X(t)$ is given by

(3.8) $$X(t) = \exp[(t-t_0)A],$$

where exp (A) is defined by

$$\exp(A) = I + \sum_{n=1}^{\infty} \frac{1}{n!} A^n .$$

The stability of the system (3.7) is directly related to the nature of the roots of the characteristic polynomial of A given by

(3.9) $$\det(A - \lambda I) = 0.$$

Since (3.9) is an nth degree polynomial equation in λ, there exists n roots to this equation (including repeated roots) which may be real or complex. Corresponding to each root is a solution of (3.7) and the n roots thus furnish n linearly independent solutions to (3.7). If λ is a real root of multiplicity k, then corresponding to λ are the k linearly independent solutions

$$e^{\lambda t};\ te^{\lambda t};\ \ldots;\ t^{k-1}e^{\lambda t}$$

while if $\lambda+i\mu$ is a complex root of multiplicity m, then so is $\lambda-i\mu$ and corresponding to these 2m roots are the 2m linearly independent solutions

$$e^{\lambda t}\cos\mu t,\ e^{\lambda t}\sin\mu t;\ te^{\lambda t}\cos\mu t,\ te^{\lambda t}\sin\mu t;\ \ldots;\ t^{m-1}\cos\mu t,\ t^{m-1}\sin\mu t.$$

Since sin t and cos t are bounded, and since for $\alpha, \beta > 0$, $\lim_{t\to\infty} t^{\alpha}e^{-\beta t} = 0$, we note that if the real parts of the eigenvalues of A (roots to (3.9)) are negative, then all solutions of (3.7) will converge to zero. When this is the case, we have the following definition.

Definition 3.4. If the real parts of the eigenvalues of A are negative, then we say that the characteristic polynomial of A is stable.

When the characteristic polynomial of A is stable, then the solutions of (3.7) are of exponential order as the following lemma illustrates.

Lemma 3.2. If the characteristic polynomial of A if stable, then all solutions of (3.7) satisfy

$$||x(t)|| \leq Me^{-\alpha t} \quad t \geq t_0, \tag{3.10}$$

where M and α are positive constants.

Proof: Let $\lambda_k = \mu_k + i\sigma_k$, $k = 1,\dots,m$, $m \leq n$, be the distinct eigenvalues of A. Since the characteristic polynomial of A is stable, there exists $\alpha > 0$ such that $\mu_k + \alpha < 0$, $k = 1,\dots m$. A solution $z_k(t)$ corresponding to λ_k is of the form

$$z_k(t) = t^r e^{\lambda k t} b_k,$$

where b_k is a constant vector having the entry 1 in one of its rows and zeros elsewhere, and where r depends on the multiplicity of λ_k. Then since $\mu_k + \alpha < 0$,

$$||z_k(t)|| e^{\alpha t} \to 0 \quad \text{as } t \to \infty$$

and thus there exist constants M_k such that

$$||z_k(t)|| \leq M_k e^{-\alpha t} \quad \text{for } t \geq t_0.$$

If x(t) is any solution to (3.7), it can be expressed as the linear combination

$$x(t) = \Sigma \beta_i z_i(t),$$

where the summation is over the n basis solutions. Then

$$||x(t)|| \leq \Sigma|\beta_i|\ ||z_i(t)||$$

$$\leq \max\ |\beta_i|\ \Sigma\ M_i\ e^{-\alpha t} \leq M\ e^{-\alpha t},$$

where $M = \max\ |\beta_i|\ \Sigma M_i$.

It is now easy to show that when the characteristic polynomial of A is stable, then (3.7) is an asymptotically stable system.

Proposition 3.3. If the characteristic polynomial of A is stable, then the system $x' = Ax$ is asymptotically stable.

Proof: By Lemma 3.2, every solution x(t) of (3.7) satisfies $||x(t)|| \leq M\ e^{-\alpha t}$ for appropriate M and α. Since the columns of X(t) are solutions, where X(t) is the fundamental matrix of (3.7), we have

$$||X(t)|| \leq Mne^{-\alpha t} \quad t \geq t_0,$$

and thus $||X(t)|| \to 0$ as $t \to \infty$ and the system is asymptotically stable by Corollary 1 to Theorem 3.1.

Thus the stability of a linear homogeneous system with constant coefficients depends solely upon the nature of the eigenvalues of A. The question now arises as to the stability of a system (3.4) in which A(t) is not constant. Proposition 3.1 insures that integrability of A(t) is sufficient to guarantee stability in the system. However, this is a rather stringent requirement and is of little practical significance in determining the stability of a given system. It is often easier to determine stability in a given system by comparing it to a known stable system. The simplest case in which this may be done successfully is when A(t) is asymptotic to a constant matrix B. The following theorem then applies

Theorem 3.2. If the characteristic polynomial of B is stable, $A(t)$ is continuous on $[0,\infty)$, and

$$\int_0^\infty ||A(t) - B||\, dt < \infty ,$$

then the system $x' = A(t)\, x$ is asymptotically stable.

Proof: The system $x' = A(t)\, x$ may be written in the form

$$x' = Bx + [A(t) - B]\, x$$

and in this manner may be treated as though it were a non-homogeneous system with constant coefficients. The solution $x(t;0,x_0)$ may then be written by Proposition 2.2 and (2.13)

$$x(t) = X(t)\, x_0 + \int_0^t X(t-s)\, [A(s) - B]\, x(s)\, ds,$$

where $X(t)$ is the fundamental matrix of the constant system $x' = B\, x$. By Lemma 3.2., there exist $M,\alpha > 0$ such that $||X(t)|| \leq Me^{-\alpha t}$ and thus

$$||x(t)||\, e^{\alpha t} \leq M\, ||x_0|| + \int_0^t M\, ||A(s)-B||\, ||x(s)||\, e^{\alpha s} ds.$$

Now applying Gronwalls' Lemma,

$$||x(t)||\, e^{\alpha t} \leq M\, ||x_0||\, \exp \int_0^t M\, ||A(s)-B||\, ds,$$

from which

$$||x(t)||\, e^{\alpha t} \leq M\, ||x_0||\, \exp \int_0^\infty M\, ||A(s)-B||\, ds$$

$$\leq K,$$

where K is a convenient constant. Hence $||x(t)|| \leq Ke^{-\alpha t}$ and if

$y(t;0,y_0)$ is any other solution,

$$||x(t) - y(t)|| \leq 2Ke^{-\alpha t} \to 0 \quad \text{as } t\to\infty$$

and thus the system $x' = A(t)\,x$ is asymptotically stable.

Theorem 3.2. may be modified in the event $A(t)$ does not differ much from B, i.e., $||A(t) - B||$ is small as follows.

<u>Theorem 3.3</u>. If the characteristic polynomial of B is stable, and $||A(t) - B|| \leq c$ on $[0,\infty)$ for sufficiently small c, then the system $x' = A(t)\,x$ is asymptotically stable.

Proof: Let $x(t;0,x_0)$ be a solution of $x' = A(t)\,x$. Then $x(t)$ may be written

$$x(t) = X(t)\,x_0 + \int_0^t X(t-s)[A(s) - B]\,x(s)\,ds,$$

where $X(t)$ is the fundamental matrix of $x' = B\,x$. Then proceeding as in the proof of Theorem 3.2, we have as before

$$||x(t)||\,e^{\alpha t} \leq M\,||x_0||\,\exp M \int_0^t ||A(s) - B||\,ds,$$

where M and α are as previously defined in Theorem 3.2. Thus

$$||x(t)|| \leq M\,||x_0||\,\exp\,(Mc-\alpha)\,t\,,$$

so that if c is sufficiently small, i.e., $c < \frac{\alpha}{M}$, then $x(t)\to 0$ as $t\to\infty$ and since $x(t)$ is an arbitrary solution, the system is asymptotically stable.

If the characteristic polynomial of B is not stable but the solutions of $x' = B\,x$ are bounded, then the following theorem holds.

<u>Theorem 3.4</u>. If all solutions of $x' = B\,x$ are bounded on $[0,\infty)$, $A(t)$ is continuous on $[0,\infty)$ and

$$\int_0^\infty ||A(t) - B||\,dt < \infty,$$

then the system $x' = A(t)\,x$ is stable.

Proof: Let $X(t)$ be the fundamental matrix of the constant system $x' = B\,x$. Then there exists $M > 0$ such that $||X(t)|| \leq M$ for $t \geq 0$. If $x(t;0,x_0)$ is a solution of $x' = A(t)\,x$, then

$$x(t) = X(t)\,x_0 + \int_0^t X(t-s)[A(s) - B]\,x(s)\,ds.$$

Thus

$$||x(t)|| \leq M\,||x_0|| + \int_0^t M\,||A(s) - B||\,||x(s)||\,ds,$$

and applying Gronwall's Lemma,

$$||x(t)|| \leq M\,||x_0|| \exp \int_0^t M\,||A(s) - B||\,ds,$$

from which

$$\begin{aligned} ||x(t)|| &\leq M\,||x_0|| \exp \int_0^\infty M\,||A(s) - B||\,ds \\ &\leq K < \infty \end{aligned}$$

where K is a suitable constant. The above shows that all solutions are bounded and hence the system $x' = A(t)\,x$ is stable by Theorem 3.1.

Theorems 3.2, 3.3, and 3.4 insure that if $||A(t) - B||$ is well behaved, then the solutions of $x' = A(t)\,x$ will resemble those of the constant system $x' = B\,x$ as $t \to \infty$ when $A(t)$ is asymptotic to B. In view of this, we might expect these results to generalize to the case when B is not constant. This is not always the case though it is possible to effect a comparison between two non-constant systems under more restrictive hypotheses. Toward this end, we prove the following lemma.

Lemma 3.3. If the system $x' = A(t)\,x$ is stable on $[t_o,\infty)$ and

$$\liminf_{t\to\infty} \int_{t_o}^{t} trA(s)\,ds > -\infty ,$$

then $X^{-1}(t)$ is bounded on $[t_o,\infty)$, where $X(t)$ is the fundamental matrix of the system.

Proof: Let $X(t)$ be the fundamental matrix of the system. Then

$$X^{-1}(t) = \frac{adj\ X(t)}{det\ X(t)}$$

and since $x' = A(t)\,x$ is stable, it follows that $X(t)$ is bounded for $t \geq t_o$ and this implies that there exists $L > 0$ such that

$$||adj\ X(t)|| \leq L \quad \text{for } t \geq t_o .$$

Then this relation and (2.8) yield

$$||X^{-1}(t)|| \leq \frac{L}{\exp \int_{t_o}^{t} tr\ A(s)\,ds}$$

and since

$$\liminf_{t\to\infty} \int_{t_o}^{t} tr\ A(s)\,ds > -\infty,$$

it follows that

$$\exp \int_{t_o}^{\infty} tr\ A(s)\,ds \geq k > 0,$$

so that

$$||X^{-1}(t)|| \leq \frac{L}{k} \quad \text{for } t \geq t_o$$

and thus $X^{-1}(t)$ is bounded.

We can now state the following comparison theorem for two non-constant systems.

<u>Theorem 3.5</u>. If the system $x' = B(t)\,x$ is stable on $[t_0,\infty)$,

$$\liminf_{t\to\infty} \int_{t_0}^{t} \operatorname{tr} B(s)\,ds > -\infty,$$

$A(t)$ is continuous on $[t_0,\infty)$ and

$$\int_{t_0}^{\infty} ||A(t) - B(t)||\,dt < \infty,$$

then the system $x' = A(t)\,x$ is stable.

Proof: Let $X(t)$ be the fundamental matrix of the system $x' = B(t)\,x$. Then the hypotheses imply that there exists $M > 0$ such that for $t \geq t_0$, $||X(t)|| \leq M$, and $||X^{-1}(t)|| \leq M$. Then if $x(t;t_0,x_0)$ is any solution to $x' = A(t)\,x$, $x(t)$ may be written

$$x(t) = X(t)\,x_0 + \int_{t_0}^{t} X(t)\,X^{-1}(s)\,[A(s) - B(s)]x(s)ds,$$

and utilizing the above we obtain

$$||x(t)|| \leq M\,||x_0|| + \int_{t_0}^{t} M^2\,||A(s) - B(s)||\;||x(s)||ds.$$

Applying Gronwall's Lemma yields

$$||x(t)|| \leq M\,||x_0||\,\exp \int_{t_0}^{t} M^2\,||A(s) - B(s)||\,ds,$$

so that

$$||x(t)|| \leq M\,||x_0||\,\exp \int_{t_0}^{\infty} M^2\,||A(s) - B(s)||ds$$

$$\leq K,$$

where K is a convenient constant. Thus all solutions of $x' = A(t)\,x$ are bounded and hence stable by Theorem 3.1.

Theorem 3.5 gives one set of conditions under which a comparison may be effected between two non-constant linear homogeneous systems and is a well known result. Another set of conditions which have been formulated by the author and which are also sufficient to effect a comparison are stated below in the following theorem.

Theorem 3.6. If the system $x' = B(t)\,x$ is stable on $[t_0,\infty)$, $A(t)$ is continuous on $[t_0,\infty)$ and

$$\int_{t_0}^{\infty} ||X^{-1}(t)\,[A(t) - B(t)]\,X(t)||\,dt < \infty$$

where $X(t)$ is the fundamental matrix of $x' = B(t)\,x$, then the system $x' = A(t)\,x$ is stable on $[t_0,\infty)$.

Proof: Let $X(t)$ be the fundamental matrix of $x' = B(t)\,x$. Then if $x(t;t_0,x_0)$ is any solution to $x' = A(t)\,x$, $x(t)$ may be written

$$x(t) = X(t)\,x_0 + \int_{t_0}^{t} X(t)\,X^{-1}(s)\,[A(s)-B(s)]x(s)\,ds.$$

From this we obtain

$$\begin{aligned} X^{-1}(t)x(t) &= x_0 + \int_{t_0}^{t} X^{-1}(s)[A(s) - B(s)]x(s)\,ds \\ &= x_0 + \int_{t_0}^{t} X^{-1}(s)[A(s) - B(s)]X(s)X^{-1}(s)\,x(s)ds \end{aligned}$$

and thus

$$||X^{-1}(t)x(t)|| \le ||x_0|| + \int_{t_0}^{t} ||X^{-1}(s)[A(s)-B(s)]X(s)||\;||X^{-1}(s)\,x(s)||\,ds,$$

so that applying Gronwall's Lemma yields

$$||X^{-1}(t)x(t)|| \leq ||x_0|| \exp \int_{t_0}^{t} ||X^{-1}(s)[A(s)-B(s)] X(s)|| \, ds.$$

Therefore,

$$||X^{-1}(t)x(t)|| \leq ||x_0|| \exp \int_{t_0}^{\infty} ||X^{-1}(s)[A(s)-B(s)] X(s)|| \, ds$$

$$\leq K < \infty$$

which shows that $X^{-1}(t)x(t)$ is bounded.

However,

$$||x(t)|| = || X(t)X^{-1}(t)x(t)|| \leq ||X(t)|| \, ||X^{-1}(t) x(t)||$$

so that $||x(t)|| \leq K \, ||X(t)||$ and since $x' = B(t)\, x$ is stable, $X(t)$ is bounded and thus $x(t)$ is bounded. Hence all solutions to $x' = A(t)\, x$ are bounded and thus stable by Theorem 3.1.

Theorems 3.2 through 3.5 provide conditions under which the stability of a given system may be determined by a comparison with a known stable system. Other conditions under which similar theorems can be established have been given by Bellman [1], and Levinson [6], as well as numerous other authors.

IV. NON-HOMOGENEOUS LINEAR SYSTEMS

The systems considered here are of the form

$$x' = A(t)\,x + b(t) \tag{4.1}$$

where $A(t)$ and $b(t)$ are continuous on $[t_0,\infty)$.

Recall that for the homogeneous system, $b(t) \equiv 0$, boundedness and stability of the system are equivalent. This is no longer true for the system (4.1) as the following example illustrates.

Example 4.1. The one-dimensional system

$$x' = 2t \quad \text{on } [0,\infty)$$

has solutions $x(t;0,x_0)$ of the form

$$x(t) = x_0 + t^2$$

which are stable yet unbounded.

The above example shows that it is possible for a non-homogeneous to be stable but unbounded. The reverse behavior, however, is not possible as the following theorem shows.

Theorem 4.1. If the system $x' = A(t)\,x + b(t)$, $A(t)$ and $b(t)$ continuous on $[t_0,\infty)$, is bounded on $[t_0,\infty)$ then it is stable on $[t_0,\infty)$.

Proof: Let $X(t)$ be the fundamental matrix of the homogeneous system $x' = A(t)\,x$. If (4.1) is bounded, then in particular, the solution $x(t;t_0,0)$ given by

$$x(t) = X(t) \int_{t_0}^{t} X^{-1}(s)\, b(s)\, ds$$

is bounded. Thus there exists $M_1 > 0$ such that $||x(t)|| \leq M_1$, $t \geq t_0$.

If $y(t;t_0,y_0)$ is any other solution to (4.1), then

$$y(t) \equiv X(t)y_0 + \int_{t_0}^{t} X(t)X^{-1}(s)\, b(s)\, ds$$

and there exists M_2 such that

$$||y(t)|| = ||X(t)\, y_0 + x(t)|| \le M_2 \quad \text{for } t \ge t_0.$$

Then

$$||X(t)\, y_0|| - ||x(t)|| \le M_2,$$

so that

$$||X(t)\, y_0|| \le M \quad t \ge t_0,$$

where $M = M_1 + M_2$. Therefore if $w(t;t_0,w_0)$ and $v(t;t_0,v_0)$ are any two solutions of (4.1) and if $\varepsilon > 0$, then for $t \ge t_0$

$$||w(t) - v(t)|| = ||X(t)\, (w_0-v_0)|| \le M\, ||w_0-v_0||$$

so that if $||w_0-v_0|| < \frac{\varepsilon}{M}$, then $||w(t)-v(t)|| < \varepsilon \quad t \ge t_0$, which shows that (4.1) is stable.

Theorem 4.1 admits the following partial converse.

<u>Theorem 4.2</u>. If the system (4.1) is stable and one solution is bounded, then all solutions are bounded.

Proof: Let $X(t)$ be the fundamental matrix of the homogeneous system $x' = A(t)\, x$. Once again the first part of the proof consists of showing that $X(t)$ is bounded.

If (4.1) is stable, then in particular, the solution $x(t;t_0,0)$ given by

$$x(t) = \int_{t_0}^{t} X(t)X^{-1}(s)b(s)\, ds$$

is stable. Therefore, if $\varepsilon > 0$, there exists $\delta > 0$ such that for $t \ge t_0$,

$$||w(t)-x(t)|| = ||X(t)w_0|| < \varepsilon$$

whenever $w(t;t_0,w_0)$ is a solution with $||w_0|| < \delta$. Letting $w_0 = \frac{\delta}{2} e^i$, where e^i is the vector whose jth component is $e^i_j = \delta_{ij}$, we obtain from the above

$$||\phi^i(t)|| < \frac{2\varepsilon}{\delta}$$

where $\phi^i(t)$ is the ith column of $X(t)$. Then since this is valid for $i = 1,\ldots,n$,

$$||X(t)|| < \frac{2n\varepsilon}{\delta} = M.$$

Now let $z(t;t_0,z_0)$ be a bounded solution and $y(t;t_0,y_0)$ any arbitrary solution of (4.1). Then since $y(t) = z(t) + [y(t) - z(t)]$,

$$\begin{aligned} ||y(t)|| &\leq ||z(t)|| + ||X(t)[y_0-z_0]|| \\ &\leq ||z(t)|| + M\,||y_0 - z_0|| \end{aligned}$$

which shows that $y(t)$ is bounded. Hence the system (4.1) is bounded.

Theorems 4.1 and 4.2 show that in a non-homogeneous linear system, boundedness is a stronger property than stability. For this reason, it is usually easier to determine stability in a system (4.1) than it is boundedness. In fact, the following proposition shows that the stability of (4.1) may be determined by simply examing its homogeneous part.

Proposition 4.1. If the system $x' = A(t)\,x$ is (asymptotically) stable on $[t_0,\infty)$, then so is the system $x' = A(t)\,x + b(t)$, where $b(t)$ is continuous on $[t_0,\infty)$.

Proof: If $x(t;t_0,x_0)$ and $y(t;t_0,y_0)$ are two solutions of $x' = A(t)\,x + b(t)$, then the difference $[x(t)-y(t)]$ satisfies the homogeneous system $x' = A(t)\,x$. Thus since the trivial solution $z(t) \equiv 0$ of the homogeneous system is stable, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $||(x_0 - y_0) - 0|| < \delta$ then

$$||x(t) - y(t) - 0|| < \varepsilon \quad t \geq t_0,$$

which is precisely the statement that the non-homogeneous system is stable. The proof is the same if the homogeneous system is, in addition, asymptotically stable.

If $A(t)$ is a constant matrix A, then (4.1) becomes

$$x' = A\,x + b(t) \tag{4.2}$$

and boundedness is easier to determine. If the characteristic polynomial of A is stable, then the following theorem holds.

Theorem 4.3. If the characteristic polynomial of A is stable, $b(t)$ is continuous and $||b(t)|| \leq K$, for $t \geq 0$, then the system (4.2) is bounded.

Proof: Let $X(t)$ be the fundamental matrix of the constant system $x' = A\,x$. By Lemma 3.2, there exist M and α such that $||X(t)|| \leq Me^{-\alpha t}$ for $t \geq 0$. If $x(t;t_0,x_0)$ is a solution of (4.2) then from

$$x(t) = X(t)x_0 + \int_0^t X(t-s)b(s)\,ds,$$

we deduce successively

$$\begin{aligned} ||x(t)|| &\leq M\,||x_0|| + \int_0^t Me^{-\alpha(t-s)}K\,ds \\ &\leq M\,||x_0|| + \frac{MK}{\alpha}(1-e^{-\alpha t}) \\ &\leq M\,||x_0|| + \frac{MK}{\alpha} \end{aligned}$$

which shows that $x(t)$ is bounded. Thus the system (4.2) is bounded.

A similar theorem holds when the solutions of $x' = A\,x$ are bounded, though the forcing term, $b(t)$, must be integrable.

Theorem 4.4. If the constant system $x' = A\,x$ is bounded on $[0,\infty)$, $b(t)$ is continuous on $[0,\infty)$ and

$$\int_0^\infty ||b(t)||\,dt < \infty ,$$

then the system $x' = A\,x + b(t)$ is bounded.

Proof: Let $X(t)$ be the fundamental matrix of $x' = A\,x$. Then there exists $M > 0$ such that $||X(t)|| \le M \quad t \ge 0$. If $x(t;0,x_0)$ is a solution to $x' = A\,x + b(t)$, then from

$$x(t) = X(t)\,x_0 + \int_0^t X(t-s)b(s)\,ds,$$

we deduce successively

$$\begin{aligned} ||x(t)|| &\le M\,||x_0|| + \int_0^t M\,||b(s)||\,ds \\ &\le M\,||x_0|| + \int_0^\infty M\,||b(s)||\,ds \\ &\le K < \infty \end{aligned}$$

where K is a convenient constant. Thus the system $x' = A\,x + b(t)$ is bounded.

The boundedness of a non-homogeneous system $x' = B(t)\,x + b(t)$ when $B(t)$ is asymptotic to a constant matrix A may be determined by effecting a comparison between it and the constant system $x' = A\,x$. The following theorem gives one instance when this may be done successfully.

Theorem 4.5. If the characteristic polynomial of A is stable, $b(t)$ is continuous and bounded on $[0,\infty)$, $B(t)$ is continuous on $[0,\infty)$ and $\int_0^\infty ||B(t) - A||\,dt < \infty$, then the system $x' = B(t)\,x + b(t)$ is bounded and asymptotically stable.

Proof: Let $X(t)$ be the fundamental matrix of the constant system $x' = A x$. Then there exist $M, \alpha > 0$ such that $||X(t)|| \leq Me^{-\alpha t}$ for $t \geq 0$. If $x(t;0,x_0)$ is a solution to $x' = B(t) x + b(t)$, then as before,

$$x(t) = X(t)x_0 + \int_0^t X(t-s)[(B(s)-A)x(s) + b(s)]\, ds,$$

from which

$$||x(t)|| \leq ||X(t)||\ ||x_0|| + \int_0^t ||X(t-s)|| \, (||B(s)-A||\ ||x(s)|| + ||b(s)||)\, ds.$$

Then since $||b(t)|| \leq N$

$$||x(t)||\, e^{\alpha t} \leq M\ ||x_0|| + \int_0^t Me^{-\alpha s}||B(s)-A||\ ||x(s)||ds + \int_0^t MNe^{-\alpha s}\, ds$$
$$\leq M\ ||x_0|| + \int_0^t Me^{-\alpha s}||x(s)||\ ||B(s)-A||ds + \int_0^\infty MNe^{-\alpha s}\, ds$$

and thus

$$||x(t)||\, e^{\alpha t} \leq L + \int_0^t Me^{-\alpha s}||x(s)||\ ||B(s)-A||\, ds,$$

where L is a convenient constant. Applying Gronwall's Lemma yields

$$||x(t)||e^{\alpha t} \leq L \exp \int_0^t M\ ||B(s)-A||\, ds,$$

so that

$$||x(t)||e^{\alpha t} \leq L \exp \int_0^\infty M\ ||B(s)-A||\, ds \leq K < \infty$$

where K is another constant. Thus for $t \geq 0$, $||x(t)|| \leq Ke^{-\alpha t} \leq K$, which shows that $x(t)$ and hence the system $x' = B(t)\,x + b(t)$ is bounded.

In addition, if $y(t;0,y_0)$ is any other solution to this system,

$$||x(t) - y(t)|| \leq ||x(t)|| + ||y(t)|| \leq 2Ke^{-\alpha t}$$

which shows that $||x(t) - y(t)|| \to 0$ as $t \to \infty$ and the system $x' = B(t)\,x + b(t)$ is also asymptotically stable.

If the characteristic polynomial of A is not assumed stable, but the system $x' = A\,x$ is bounded, then a similar theorem holds.

Theorem 4.6. If the system $x' = A\,x$ is bounded on $[0,\infty)$, $b(t)$ and $B(t)$ are continuous on $[0,\infty)$,

$$\int_0^\infty ||b(s)||\,ds < \infty, \text{ and}$$

$$\int_0^\infty ||B(s) - A||\,ds < \infty\,,$$

then the system $x' = B(t)x + b(t)$ is bounded.

Proof: Let $X(t)$ be the fundamental matrix of the constant system $x' = A\,x$. Then there exists $M > 0$ such that $||X(t)|| \leq M$ for $t \geq 0$. If $x(t;0,x_0)$ is any solution to $x' = B(t)\,x + b(t)$, then from

$$x(t) = X(t)x_0 + \int_0^t X(t-s)[(B(s)-A)x(s)+b(s)]\,ds,$$

we have successively

$$||x(t)|| \leq M\,||x_0|| + \int_0^t M\,||B(s)-A||\,||x(s)||\,ds$$
$$+ \int_0^t M\,||b(s)||\,ds$$

$$\leq M\,||x_0|| + \int_0^t M\,||B(s)-A||\;||x(s)||\,ds$$

$$+ \int_0^\infty M\,||b(s)||\,ds$$

$$\leq K + \int_0^t M||B(s)-A||\;||x(s)||\,ds,$$

where K is a suitable constant. Applying Gronwall's Lemma yields

$$||x(t)|| \leq K \exp \int_0^t M\,||B(s)-A||\,ds$$

$$\leq K \exp \int_0^\infty M\,||B(s)-A||\,ds \leq L < \infty$$

where L is another constant. Thus x(t) and hence the system $x' = B(t)\,x + b(t)$ are bounded.

In general, if A(t) is not constant, Theorems 4.3 through 4.6 need no longer apply. Thus if a homogeneous system $x' = A(t)\,x$ is bounded (stable) and

$$\int_0^\infty ||b(t)||\,dt < \infty,$$

it need not follow that the non-homogeneous system $x' = A(t)\,x + b(t)$ is bounded. The same applies if the homogeneous system is asymptotically stable. A similar problem was encountered in attempting to effect a stability comparison between two non-constant linear homogeneous systems. There it was noted that additional hypotheses were required to insure the stability of the system $x' = B(t)\,x$ when B(t) is asymptotic to a non-constant matrix A(t) and the system $x' = A(t)\,x$ is stable. One reason for this is that stability, as defined in the sense of Lyapunov, is quite delicate and often may not be maintained even when small perturbations are introduced. This leads to the concepts of uniform and restrictive stability, which are discussed in section V.

V. UNIFORM AND RESTRICTIVE STABILITY

Since Lyapunov stability does not suffice to insure certain boundedness and comparison properties, we are led to a consideration of stronger forms of stability. One of these is uniform stability, defined as follows.

Definition 5.1. Let $x(t;t_0,x_0)$ be a solution of the system $x' = f(t,x)$, where $f(t,x)$ is defined and continuous in the region $\Gamma = \{(t,x)\colon \ t_0 \leq t < \infty, \ \ ||x|| < \xi \}$. Then $x(t)$ is said to be uniformly stable if it satisfies (3.2i,ii, and iii) and in addition (5.1) given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon,f) > 0$ such that every solution $y(t;t_0,y_0)$ with $||x(t_1) - y(t_1)|| < \delta$ for some $t_1 \geq t_0$ satisfies $||x(t) - y(t)|| < \varepsilon$ for $t \geq t_1$.

Comparing the definitions, we note that uniform stability is similar to Lyapunov stability. However, there is an important distinction between the two. A solution $x(t)$ is stable in the sense of Lyapunov if any other solution which is initially close to $x(t)$ remains in an ε-neighborhood of $x(t)$ for $t \geq t_0$; at later times, i.e., $t > t_0$; other solutions may come arbitrarily close but need not remain there. If $x(t)$ is uniformly stable, once a solution enters a prescribed small neighborhood of $x(t)$ for any $t_1 \geq t_0$, it must remain there for all $t \geq t_1$. Thus, in uniform stability, δ no longer depends on t_0.

The above remarks show that a uniformly stable solution is stable. If all solutions to a system are uniformly stable, we say that the system is uniformly stable. For linear homogeneous systems of the form

$$(5.2) \qquad x' = A(t)\,x,$$

if one solution to the system is uniformly stable, then the system is uniformly stable. Uniform stability for a system (5.2) may be characterized by the behavior of the fundamental matrix of the system as the following theorem shows.

Theorem 5.1. The system $x' = A(t)\,x$ is uniformly stable iff there exists $M > 0$ such that $||X(t)X^{-1}(s)|| \leq M \quad t \geq s \geq t_0$, where $X(t)$ is the fundamental matrix of the system.

Proof: Assume that the system $x' = A(t)\,x$ is uniformly stable, so that the trivial solution $z(t) \equiv 0$ is uniformly stable. Then if $\varepsilon > 0$ there exists $\delta > 0$ such that if $||x(t_1)|| < \delta$ for some $t_1 \geq t_0$, then $||x(t)|| < \varepsilon$ for $t \geq t_1$. Let $y(t;t_0,y_0)$ be a solution of $x' = A(t)\,x$. Then $y(t)$ may be written

$$y(t) = X(t)\,X^{-1}(t_1)\,y(t_1).$$

By choosing $y(t)$ to be that solution satisfying $y(t_1) = \frac{\delta}{2} e^i$, where e^i is as in the proof of Theorem 3.1, we have $||y(t_1)|| = \frac{\delta}{2} < \delta$ so that

$$\frac{\delta}{2}||X(t)X^{-1}(t_1)e^i|| < \varepsilon \qquad t \geq t_1,$$

or equivalently,

$$||Z^i(t,t_1)|| < \frac{2\varepsilon}{\delta} \qquad t \geq t_1$$

where $Z^i(t,t_1)$ is the ith column of the matrix $X(t)X^{-1}(t_1)$. Repeating the above procedure for $i = 1,\ldots,n$ yields

$$||X(t)X^{-1}(t_1)|| < \frac{2n\varepsilon}{\delta} \qquad t \geq t_1.$$

Finally, since this procedure is valid for any $t_1 \geq t_0$ and the bound is independent of t_1, we obtain

$$||X(t)X^{-1}(t_1)|| < \frac{2n\varepsilon}{\delta} = M, \ t \geq t_1 \geq t_0.$$

Conversely, if $||X(t)\ X^{-1}(t_1)|| \leq M$ for $t \geq t_1 \geq t_0$, and $x(t;t_0,x_0)$ and $y(t;t_0,y_0)$ are any two solutions of $x' = A(t)\ x$, then for $t \geq t_1 \geq t_0$

$$\begin{aligned} ||x(t)-y(t)|| &= ||X(t)X^{-1}(t_1)[x(t_1)-y(t_1)]|| \\ &\leq M\ ||x(t_1) - y(t_1)|| \end{aligned}$$

so that if $\varepsilon > 0$, $||x(t_1) - y(t_1)|| < \frac{\varepsilon}{M}$ $t_1 \geq t_0$ implies $||x(t) - y(t)|| < \varepsilon$ $t \geq t_1$ which shows that the system is uniformly stable.

From Theorem 5.1, we obtain the following simple proposition.

Proposition 5.1. If the system $x' = A\ x$, where A is constant, is stable on $[0,\infty)$, then it is uniformly stable on $[0,\infty)$.

Proof: Let $X(t)$ be the fundamental matrix of $x' = A\ x$. Then there exists $M > 0$ such that $||X(t)|| \leq M$ for $t \geq 0$. Then by (2.13),

$$||X(t-s)|| = ||X(t)\ X^{-1}(s)|| \leq M \quad \text{for } t \geq s \geq 0$$

and the system is thus uniformly stable.

Recall that in sections III and IV, several boundedness and comparison properties which hold when $A(t) = A$ is constant, need not apply when $A(t)$ is not constant. However, Proposition 5.1 shows that in the case $A(t)$ is constant, the systems considered are actually uniformly stable, rather than just stable. This leads to the question of whether or not uniform stability of the non-constant system $x' = A(t)\ x$ is sufficient to insure these boundedness and comparison properties. The following theorems provide an affirmative answer to this question.

Theorem 5.2. If the system $x' = A(t)\,x$ is uniformly stable on $[t_0,\infty)$, $B(t)$ is continuous on $[t_0,\infty)$ and $\int_{t_0}^{\infty} ||B(t)-A(t)||\, dt$ then the system $x' = B(t)\,x$ is uniformly stable.

Proof: Let $X(t)$ be the fundamental matrix of the system $x' = A(t)\,x$. Then $||X(t)\,X^{-1}(s)|| \leq M$ for $t \geq s \geq t_0$. Let $\varepsilon > 0$ and fix $s \geq t_0$. If $x(t;t_0,x_0)$ and $y(t;t_0,y_0)$ are two solutions of the system $x' = B(t)x$, then since $B(t) = A(t) + [B(t) - A(t)]$, we have for $t \geq s$,

$$x(t)-y(t) = X(t)X^{-1}(s)[x(s)-y(s)] + \int_s^t X(t)X^{-1}(\tau)[B(\tau)-A(\tau)][x(\tau)-y(\tau)]\, d\tau$$

and thus

$$\begin{aligned} ||x(t)-y(t)|| &\leq ||X(t)X^{-1}(s)||\; ||x(s)-y(s)|| + \int_s^t ||X(t)X^{-1}(\tau)||\; ||B(\tau)-A(\tau)||\,||x(\tau)-y(\tau)||\,d\tau \\ &\leq M\, ||x(s)-y(s)|| + \int_{t_0}^t M\, ||B(\tau)-A(\tau)||\; ||x(\tau)-y(\tau)||\, d\tau. \end{aligned}$$

Applying Gronwall's Lemma yields

$$||x(t)-y(t)|| \leq M\, ||x(s)-y(s)|| \exp \int_{t_0}^t M\, ||B(\tau)-A(\tau)||\, d\tau,$$

from which

$$||x(t)-y(t)|| \leq M\, ||x(s)-y(s)|| \exp \int_{t_0}^{\infty} M\, ||B(\tau)-A(\tau)||\, d\tau.$$

Therefore,

$$||x(t)-y(t)|| \leq L\ ||x(s)-y(s)||$$

where L is a convenient constant. The above relation is valid for $t \geq s \geq t_0$, so that if $||x(s)-y(s)|| < \frac{\varepsilon}{L}$ $s \geq t_0$, then $||x(t)-y(t)|| < \varepsilon$ for $t \geq s$ which shows that the system $x' = B(t)\ x$ is uniformly stable.

<u>Corollary 1</u>. Under the same hypotheses as Theorem 5.2, the system $x' = B(t)\ x + b(t)$, where $b(t)$ is continuous on $[t_0,\infty)$ is uniformly stable on $[t_0,\infty)$.

Proof: The proof follows from the fact that under the hypotheses, the system $x' = B(t)\ x$ is uniformly stable and that if $x(t)$ and $y(t)$ are two solutions to the non-homogeneous system $x' = B(t)\ x + b(t)$, then the difference $x(t) - y(t)$ is a solution to the non-homogeneous system $x' = B(t)\ x$.

When the homogeneous system is uniformly stable, then boundedness of the non-homogeneous system may be determined as for the constant case considered earlier in Section IV.

<u>Theorem 5.3</u>. If the system $x' = A(t)\ x$ is uniformly stable on $[t_0,\infty)$, $b(t)$ is continuous on $[t_0,\infty)$ and $\int_0^\infty ||b(s)||\ ds < \infty$, then the system $x' = A(t)\ x + b(t)$ is bounded on $[t_0,\infty)$.

Proof: Let $x(t;t_0,x_0)$ be a solution to $x' = A(t)\ x + b(t)$. Then

$$x(t) = X(t)x_0 + \int_{t_0}^{t} X(t)X^{-1}(s)b(s)\ ds,$$

where $X(t)$ is the fundamental matrix of the homogeneous system $x' = A(t)\ x$. Since $||X(t)\ X^{-1}(s)|| \leq M$ for $t \geq s \geq t_0$,

$$||x(t)|| \leq M\,||x_0|| + \int_{t_0}^{t} M\,||b(s)||\,ds$$

and thus

$$||x(t)|| \leq M\,||x_0|| + \int_{t_0}^{\infty} M\,||b(s)||\,ds \leq L < \infty$$

where L is a convenient constant. Thus the system $x' = A(t)\,x + b(t)$ is bounded.

It is also possible to determine boundedness of a non-homogeneous system through comparison with a uniformly stable system.

Proposition 5.2. If the system $x' = A(t)\,x$ is uniformly stable on $[t_0,\infty)$, $B(t)$ and $b(t)$ are continuous on $[t_0,\infty)$, $\int_{t_0}^{\infty}||b(t)||\,dt < \infty$, and $\int_{t_0}^{\infty}||B(t) - A(t)||\,dt < \infty$, then the system $x' = B(t)\,x + b(t)$ is bounded on $[t_0,\infty)$.

Proof: By Theorem 5.2, the hypotheses imply that the homogeneous system $x' = B(t)\,x$ is uniformly stable on $[t_0,\infty)$. The result then follows from Theorem 5.3.

Thus certain boundedness and comparison properties established for constant systems may be generalized to non-constant systems which are uniformly stable.

We now introduce the concept of restrictive stability.

Definition 5.2. The linear homogeneous system $x' = A(t)\,x$ is said to be restrictively stable if it is stable and its adjoint system $x' = -A(t)^T\,x$ is stable.

We note that restrictive stability applies to a system, rather than a particular solution and is defined for linear homogeneous systems only. The following theorem characterizes restrictive stability in terms of the behavior of the fundamental matrix of the system.

Theorem 5.4. The system $x' = A(t)\, x$ is restrictively stable on $[t_0,\infty)$ iff there exists $M > 0$ such that $||X(t)|| \leq M$ and $||X^{-1}(t)|| \leq M$ for $t \geq t_0$, where $X(t)$ is the fundamental matrix of the system $x' = A(t)x$.

Proof: The result follows by Theorem 3.1, noting that $[X^{-1}(t)]^T$ is the fundamental matrix of the adjoint system $x' = -A(t)^T x$.

From Theorem 5.4, we see that a restrictively stable system $x' = A(t)\, x$ is uniformly stable since

$$||X(t)X^{-1}(s)|| \leq ||X(t)||\ ||X^{-1}(s)|| \leq M^2 \quad s,t \geq t_0$$

but not necessarily conversely. Also, since $-[-A(t)^T]^T = A(t)$, it follows that if $x' = A(t)\, x$ is restrictively stable, then its adjoint system is restrictively stable. The following propositions give conditions which suffice for a system to be restrictively stable.

Proposition 5.3. If $\int_{t_0}^{\infty} ||A(t)||\, dt < \infty$, and $A(t)$ is continuous on $[t_0,\infty)$, then the system $x' = A(t)\, x$ is restrictively stable on $[t_0,\infty)$.

Proof: Proposition 3.2 shows that under the hypotheses, the system $x' = A(t)\, x$ is stable. Since $||A(t)|| = ||-A(t)^T||$, it follows that the adjoint system is stable.

Proposition 5.4. If the system $x' = A(t)\, x$ is stable on $[t_0,\infty)$ and $A(t)$ is continuous on $[t_0,\infty)$, and $\liminf_{t\to\infty} \int_{t_0}^{t} \operatorname{tr} A(s)\, ds > -\infty$, then it is restrictively stable.

Proof: Let $X(t)$ be the fundamental matrix of $x' = A(t)\,x$. Then $X(t)$ is bounded for $t \geq t_0$ since the system is stable. By Lemma 3.3, $\liminf_{t\to\infty} \int_{t_0}^{t} \operatorname{tr} A(s)\,ds > -\infty$ implies that $X^{-1}(t)$ is bounded for $t \geq t_0$. Hence the system $x' = A(t)\,x$ is restrictively stable by Theorem 5.4.

Proposition 5.5. If the adjoint system $x' = -A(t)^T x$ is stable on $[t_0,\infty)$, $A(t)$ is continuous on $[t_0,\infty)$, and $\limsup_{t\to\infty} \int_{t_0}^{t} \operatorname{tr} A(s)\,ds < \infty$, then the system $x' = A(t)\,x$ is restrictively stable on $[t_0,\infty)$.

Proof: Let $X(t)$ be the fundamental matrix of $x' = A(t)\,x$. Then since the adjoint system is stable, $X^{-1}(t)$ is bounded for $t \geq t_0$. From the relation $X(t) = \det X(t) \operatorname{adj} X^{-1}(t)$, we obtain by (2.8)

$$X(t) = \exp \int_{t_0}^{t} \operatorname{tr} A(s)\,ds \operatorname{adj} X^{-1}(t)$$

and since $X^{-1}(t)$ is bounded, $\operatorname{adj} X^{-1}(t)$ is bounded and thus

$$\begin{aligned} \|X(t)\| &\leq M \exp \int_{t_0}^{t} \operatorname{tr} A(s)\,ds \\ &\leq M \limsup_{t\to\infty} \int_{t_0}^{t} \operatorname{tr} A(s)\,ds \leq L < \infty \end{aligned}$$

where M and L are suitable constants. Hence $X(t)$ is bounded and the system $x' = A(t)\,x$ is restrictively stable by Theorem 5.4.

An interesting result concerning the well known second order Hill's Equation

$$u'' + a(t)\,u = 0 \quad \text{on } [0,\infty), \tag{5.3}$$

where $a(t)$ is real valued and continuous on $[0,\infty)$ and $u(t)$ is a scalar valued function, may be obtained through the use of stability theory. The vector analogue of (5.3) is the two-dimensional system

(5.4)
$$y' = z$$
$$z' = -a(t)\, y$$

which is in the form (5.2) with $x = \begin{pmatrix} y \\ z \end{pmatrix}$ and $A(t) = \begin{pmatrix} 0 & 1 \\ -a(t) & 0 \end{pmatrix}$.

We note that, since tr $A(t) \equiv 0$, if the system (5.4) is stable, then it will be restrictively stable, and thus uniformly stable. It is now possible to give the following result: If $\int_0^\infty |a(t)|\, dt < \infty$, then there exist unbounded solutions of (5.3). This may be proved using stability theory as follows. Assume that all solutions to (5.3) are bounded. Then (5.4) is a stable system and hence uniformly stable by the above remarks. Theorem 5.2 then asserts that if

$$B(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then the system $x' = B(t)\, x$ is uniformly stable and hence bounded since $\int_0^\infty ||B(t) - A(t)||\, dt = \int_0^\infty |a(t)|\, dt < \infty$. But this is a contradiction since

$$x(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$$

is a solution to $x' = B(t)x$. Thus all solutions to (5.3) are bounded.

Restrictive stability may also be characterized in terms of the reducibility of the system. Toward this end, we have the following definition.

Definition 5.3. The system $x' = A(t)\,x$ is said to be reducible if there exists a matrix $L(t)$ whose elements are differentiable functions which is bounded together with $L^{-1}(t)$ on $[t_0,\infty)$ such that

(5.5) $$L^{-1}(t)\,A(t)\,L(t) - L^{-1}(t)\,L'(t) = B$$

where B is a constant matrix. If $B = 0$, the zero matrix, then we say that the system is reducible to zero.

The relation (5.5) is often expressed by saying that $A(t)$ is kinematically similar to the constant matrix B. If both $L(t)$ and $A(t)$ are constant, then (5.5) becomes

(5.6) $$L^{-1}\,A\,L = B$$

which is the ordinary similarity relation.

If a linear homogeneous system is reducible, then the transformation $x = L(t)\,y$ reduces the system $x' = A(t)\,x$ to the constant system $y' = B\,y$, while if $x' = A(t)\,x$ is reducible to zero, then this transformation yields the system $y' = 0$.

The following theorem characterizes restrictive stability in terms of the reducibility of the system.

Theorem 5.5. The system $x' = A(t)\,x$ is restrictively stable on $[t_0,\infty)$ iff it is reducible to zero on $[t_0,\infty)$.

Proof: If $x' = A(t)\,x$ is restrictively stable, then there exists $M > 0$ such that $||X(t)|| \leq M$ and $||X^{-1}(t)|| \leq M$ for $t \geq t_0$, where $X(t)$ is the fundamental matrix of the system. Then since $X(t)$ satisfies $X(t) = A(t)\,X$, it follows that

$$X^{-1}(t)\,A(t)\,X(t) - X^{-1}(t)\,X'(t) \equiv 0$$

and that the system $x' = A(t)\,x$ is reducible to zero by taking $L(t) = X(t)$.

Conversely, assume that (5.5) is satisfied for $B = 0$. Then from (5.5), $L(t)$ satisfies

$$L' = A(t) L$$

and since $L(t)$ is non-singular, it follows that $L(t)$ is a basis matrix for the system $x' = A(t) x$. Hence $X(t) = L(t) W$, where $X(t)$ is the fundamental matrix and W is a constant non-singular matrix. Thus $||X(t)|| \leq ||L(t)|| \, ||W||$ and $||X^{-1}(t)|| \leq ||L^{-1}(t)|| \, ||W^{-1}||$ which shows that both $X(t)$ and $X^{-1}(t)$ are bounded. Thus the system $x' = A(t) x$ is restrictively stable by Theorem 5.4.

Uniform stability is also related to reducibility as the following theorem illustrates.

<u>Theorem 5.6</u>. If the system $x' = A(t) x$ is stable and reducible on $[0,\infty)$, then it is uniformly stable on $[0,\infty)$.

Proof: Let $X(t)$ be the fundamental matrix of $x' = A(t) x$ and let $L(t)$ be as in Definition 5.3. Then the transformation $y(t) = L^{-1}(t) x(t)$ reduces $x' = A(t) x$ to the system $y' = B y$, where B is constant. Also, since $L^{-1}(t) X(t)$ is non-singular and satisfies

$$(L^{-1} X)' = B (L^{-1} X),$$

it follows that $L^{-1}(t) X(t)$ is a basis matrix for $y' = B y$. Then $Y(t) = L^{-1}(t) X(t) W$, where $Y(t)$ is the fundamental matrix of $y' = B y$ and W is a constant non-singular matrix. Since $x' = A(t) x$ is stable, $X(t)$ is bounded and by definition, both $L(t)$ and $L^{-1}(t)$ are bounded. Thus there exists $M > 0$ such that $||X(t)|| \leq M$, $||L(t)|| \leq M$, and $||L^{-1}(t)|| \leq M$, so that $||Y(t)|| \leq M^2 \, ||W||$, which shows that $Y(t)$ is bounded and thus that the system $y' = B y$ is stable.

Then by Proposition 5.1, $y' = B\,y$ is uniformly stable and thus there exists $N > 0$ such that $||Y(t)\,Y^{-1}(s)|| \leq N$ for $t \geq s \geq 0$. Then since

$$X(t) = L(t)\,Y(t)\,W^{-1}$$

we have for $t \geq s \geq 0$

$$\begin{aligned} ||X(t)X^{-1}(s)|| &\leq ||L(t)||\ ||Y(t)Y^{-1}(s)||\ ||L^{-1}(t)|| \\ &\leq M^2\,N \end{aligned}$$

and thus $x' = A(t)\,x$ is uniformly stable by Theorem 5.1.

The above remarks relate the connection between uniform and restrictive stability in a system and the reducibility of that system. Also, we have seen that uniformly stable systems and restrictively stable systems satisfy certain boundedness and comparison properties that need not necessarily hold for stable systems. In Section VI, we shall show that a similar behavior is exhibited when the systems considered are non-linear.

VI. NONLINEAR SYSTEMS

The systems considered here are non-linear mixed systems of the form

$$(6.1) \qquad x' = A(t)\,x + f(t,x)$$

where $A(t)$ is continuous on $[0,\infty)$ and $f(t,x)$ satisfies certain appropriate conditions. Such systems are extremely important since the general first order normal system

$$(6.2) \qquad x' = g(t,x)$$

can often be put in the form (6.1) by taking $f(t,x) = g(t,x) - A(t)$ where $A(t)$ depends on the form of $g(t,x)$. Conditions on $f(t,x)$ are thus sought which insure stability in the non-linear system (6.1) when it is known that the linear portion

$$(6.3) \qquad x' = A(t)\,x$$

is stable. When $A(t) = A$ is constant, then (6.1) becomes

$$(6.4) \qquad x' = A\,x + f(t,x)$$

and stability is easier to determine.

Consider the following conditions on $f(t,x)$.

(6.5i) $f(t,x)$ is continuous in the region

$$\Gamma = \{(t,x) : 0 \leq t < \infty,\ ||x|| < \xi\}$$

(6.5ii) $||f(t,x)|| = o(||x||)$ uniformly in t as $||x|| \to 0$, i.e., there exists a function $u(r) > 0$ such that $u(r) \to 0$ as $r \to 0$ and

$$||f(t,x)|| \leq ||x||\, u(||x||).$$

Condition (6.5i) assures existence of solutions for small $||x||$ but not uniqueness. Note also that (6.5ii) implies that $z(t) \equiv 0$ is a solution to (6.4).

When $f(t,x)$ satisfies (6.5i,ii), we might heuristically reason that since $f(t,x)$ is small when $||x||$ is near zero, the system (6.4) might be expected to behave in a manner similar to $x' = A x$. Such reasoning is correct as the following theorem illustrates.

Theorem 6.1. If the characteristic polynomial of A is stable and $f(t,x)$ satisfies (6.5i, ii) then the solution $z(t) \equiv 0$ of the system $x' = A x + f(t,x)$ is asymptotically stable.

Proof: We first show that solutions $x(t;0,x_0)$ exist and are defined on $[0,\infty)$ when $||x_0||$ is small. Let $X(t)$ be the fundamental matrix of the constant system $x' = A x$. Then $X(t)$ satisfies for $t \geq 0$ $||X(t)|| \leq M e^{-\alpha t}$ for suitable M and α. The solution $x(t;0,x_0)$ of $x' = A x + f(t,x)$ satisfies (where it exists)

$$x(t) = X(t) x_0 + \int_0^t X(t-s) f(s,x(s)) ds,$$

so that

$$||x(t)||e^{\alpha t} \leq M ||x_0|| + \int_0^t Me^{\alpha s}||f(s,x(s))|| ds.$$

The above inequality is valid for t $[0,t_1)$ where $||x(t)|| < \xi$ for t $[0,t_1)$, assuming $||x_0|| < \xi$. Then from (6.5ii), given $\varepsilon > 0$, there exists $\delta > 0$ such that for $t \geq 0$,

$$||f(t,x)|| \leq \varepsilon||x||$$

whenever $||x|| < \delta$.

If we take $||x_0|| < \delta$, then by continuity of $x(t)$, there exists $t_2 > 0$ such that $||x(t)|| < \delta$ for $0 \leq t < t_2$. Therefore for $0 \leq t < T$, where $T = \min(t_1, t_2)$, we have

$$||x(t)||\, e^{\alpha t} \leq M\, ||x_0|| + M\varepsilon \int_0^t e^{\alpha s} ||x(s)||\, ds.$$

Applying Gronwall's Lemma to the above yields

$$||x(t)|| e^{\alpha t} \leq M ||x_0|| \exp \int_0^t M\varepsilon\, ds$$

and thus for $0 \leq t < T$

$$||x(t)|| \leq M\, ||x_0|| \exp(M\varepsilon - \alpha)t.$$

Choosing $\varepsilon < \frac{\alpha}{M}$ yields

$$||x(t)|| \leq M\, ||x_0|| \qquad 0 \leq t \leq T$$

so that if $||x_0|| < \frac{\delta}{2M}$,

$$||x(t)|| < \frac{\delta}{2} < \delta \qquad 0 \leq t \leq T.$$

Since $f(t,x)$ is defined and continuous for $t \geq 0$ and $||x|| < \xi$, the above relation implies that the solution $x(t;0,x_0)$ can be extended interval by interval preserving the above bound. Thus, any solution $x(t;0,x_0)$ with $||x_0|| < \frac{\delta}{2M}$ exists on $[0,\infty)$ and satisfies $||x(t)|| < \frac{\delta}{2}$ for $t \geq 0$. Since δ can be taken arbitrarily small, this implies that the solution $z(t) \equiv 0$ of $x' = A\,x + f(t,x)$ is stable. Furthermore,

$$||x(t)|| \leq M\, ||x_0|| \exp\,(M\varepsilon - \alpha)t$$

where $\varepsilon < \frac{\alpha}{M}$ implies that $||x(t)|| \to 0$ as $t \to \infty$ and thus the trivial solution is asymptotically stable.

Coddington and Levinson [3] show that the hypotheses of Theorem 6.1 may be weakened somewhat. Instead of requiring that $f(t,x)$ satisfy (6.5ii) it is sufficient to assume instead that for some $K > 0$,

$$||f(t,x)|| \leq K\ ||x|| \quad t \geq 0$$

for all small $||x||$, and that given $\varepsilon > 0$ there exist δ, T such that

$$||f(t,x)|| \leq \varepsilon\ ||x||$$

for $||x|| < \delta$ and $t \geq T$.

Theorem 6.1 may then be applied to a system of the form

$$x' = A\,x + B(t)\,x + g(t,x) \tag{6.6}$$

where $B(t) \to 0$ as $t \to \infty$ and $||g(t,x)|| = o(||x||)$.

Other conditions on $f(t,x)$ which suffice for theorems similar to Theorem 6.1 may be found in [1] and [6].

When $A(t)$ is not constant in (6.1), then Theorem 6.1 need no longer necessarily hold. However, if the linear portion $x' = A(t)\,x$ of (6.1) is uniformly stable, then we have the following theorem.

Theorem 6.2. If the linear system $x' = A(t)\,x$ is uniformly stable and $f(t,x)$ satisfies

(6.7i) $\quad f(t,x)$ defined and continuous for $t \geq 0$,
$||x|| < \xi$ and

(6.7ii) $\quad ||f(t,x)|| \leq a(t)\ ||x||$, where $a(t) > 0$ and

$$\int_0^\infty a(t)\,dt < \infty,$$

then the solution $z(t) \equiv 0$ of $x' = A(t)\,x + f(t,x)$ is uniformly stable on $[0,\infty)$.

Proof: Let $X(t)$ be the fundamental matrix of the linear system $x' = A(t)\,x$. Then there exists $M > 0$ such that $||X(t)\,X^{-1}(s)|| \leq M$ for $t \geq s \geq 0$. As in the proof of Theorem 6.1, it can be shown that any solution $x(t;0,x_0)$ of $x' = A(t)\,x + f(t,x)$ exists and is defined on $[0,\infty)$. Then assuming this, the solution $x(t;0,x_0)$ can be written

$$x(t) = X(t)X^{-1}(s)x(s) + \int_0^t X(t)X^{-1}(\tau)f(\tau,x(\tau))\, d\tau,$$

where $s \geq 0$ is fixed, so that

$$||x(t)|| \leq M||x(s)|| + \int_0^t M\, a(\tau)||x(\tau)||d\tau.$$

Applying Gronwall's Lemma yields

$$||x(t)|| \leq M||x(s)|| \exp \int_0^t M\, a(\tau)\, d\tau$$

and thus

$$||x(t)|| \leq M||x(s)|| \exp \int_0^\infty M\, a(\tau)\, d\tau \leq K||x(s)||$$

where K is a convenient constant. The above relation is valid for any $s \geq 0$ so that given $\varepsilon > 0$, $||x(s)|| < \frac{\varepsilon}{K}$ for some $s \geq 0$ implies $||x(t)|| < \varepsilon$ for all $t \geq s$ so that the trivial solution $z(t) = 0$ of $x' = A(t)\, x + f(t,x)$ is uniformly stable.

Corollary 1. Under the same hypothesis as Theorem 6.2, if the linear system $x' = A(t)\, x$ is in addition asymptotically stable, then so is the trivial solution $z(t) \equiv 0$ of $x' = A(t)\, x + f(t,x)$.

Proof: The hypotheses imply that $z(t) \equiv 0$ is a uniformly stable solution of $x' = A(t)\, x + f(t,x)$. If the linear system is in addition asymptotically stable, then $||X(t)|| \to 0$ as $t \to \infty$ where $X(t)$ is the fundamental matrix of the linear system. Therefore, if $x(t;0,x_0)$ is any solution to $x' = A(t)\, x + f(t,x)$, then given $\varepsilon > 0$ there exists T such that $||X(t)|| < \frac{\varepsilon}{x_0}$ for $t \geq T$. Then from

$$x(t) = X(t)x_0 + \int_0^t X(t)X^{-1}(s)f(s,x(s))ds,$$

we have

$$||x(t)|| \leq ||X(t)||\ ||x_0|| + \int_0^t M\, a(s)\ ||x(s)||\ ds,$$

where $||X(t)\ X^{-1}(s)|| \le M$ for $t \ge s \ge 0$. Then for $t \ge T$ we obtain

$$||x(t)|| \le \varepsilon + M \int_0^t a(s)\ ||x(s)||\ ds$$

and applying Gronwall's Lemma,

$$||x(t)|| \le \varepsilon \exp \int_0^t M\ a(s)\ ds,$$

so that
$$||x(t)|| \le \varepsilon \exp \int_0^\infty M\ a(s)\ ds \le \varepsilon K$$

where K is a suitable constant. Since $\varepsilon > 0$ is arbitrary and K is independent of ε and T, we conclude that $||x(t)|| \to 0$ as $t \to \infty$ and that the trivial solution $z(t) \equiv 0$ of $x' = A(t)\ x + f(t,x)$ is uniformly and asymptotically stable.

From the proof of Theorem 6.1, we note that when the linear system $x' = A(t)\ x$ is uniformly stable and $f(t,x)$ satisfies (6.7i,ii), then all solutions of (6.1) remain bounded. However, in the event the linear system is restrictively stable, then the solutions of (6.1) are actually bounded by those of the linear system as the following theorem illustrates.

Theorem 6.3. If the system $x' = A(t)\ x$ is restrictively stable and $f(t,x)$ satisfies (6.7i,ii), then every solution $x(t;0,x_0)$ of $x' = A(t)\ x + f(t,x)$ satisfies $x(t) = O(||X(t)||)$, i.e., $||x(t)|| \le K\ ||X(t)||\ ||x_0||$ where $K > 0$ is constant.

Proof: Let $X(t)$ be the fundamental matrix of the linear system $x' = A(t)\ x$. Then there exists $M > 0$ such that $||X(t)|| \le M$ and $||X^{-1}(t)|| \le M$ for $t \ge 0$. Let $x(t;0,x_0)$ be a solution of $x' = A(t)\ x + f(t,x)$. Then from

$$x(t) = X(t)x_0 + \int_0^t X(t)\ X^{-1}(s)\ f(s,x(s))\ ds,$$

we obtain

$$\frac{||x(t)||}{||X(t)||} \leq ||x_0|| + M \int_0^t a(s)\ ||x(s)||\ ds$$

$$\leq ||x_0|| + M^2 \int_0^t a(s) \frac{||x(s)||}{||X(s)||}\ ds.$$

Applying Gronwall's Lemma yields

$$\frac{||x(t)||}{||X(t)||} \leq ||x_0|| \exp \int_0^t M^2 a(s)\ ds$$

$$\leq ||x_0|| \exp \int_0^\infty M\ a(s)\ ds \leq K||x_0||,$$

where K is a convenient constant. Thus $||\ x(t))|| \leq K\ ||x_0||\ \ ||X(t)||$ which completes the proof.

Theorem 6.3 shows that when the linear system is restrictively stable, the contribuition of the non-linear term f(t,x) to the solutions of (6.1) is small. Golomb [4] has given conditions under which boundedness and stability may be determined in systems of the form in which f(t,x) satisfies

(6.8i) f(t,x) is continuous in x a.e. for $t \geq 0$ and is integrable in t for fixed x and

(6.8ii) $||f(t,x)|| \leq a(t)\ u(||x||)$ for $t \geq 0$, where a(t) is positive and integrable and u(r) defined for $r > 0$ is a positive non-decreasing continuous function which is submultiplicative, i.e.,

$u(r\ s) \leq u(r)\ u(s)$,

without requiring uniform stability in the linear system, essentially generalizing Theorems 6.2 and 6.3. Other evaluations of solutions to (6.1) may be found in the same paper.

VII. CONCLUSIONS

We have considered stability in the sense of Lyapunov for systems of ordinary differential equations with particular emphasis on linear systems. The theory for these systems is quite advanced today and in Sections III, IV, and V we have considered several of the more important results. Though by no means exhaustive, much of the material in these sections comes from varied sources and it is felt that the presentation there offers a more complete treatment than is usually found elsewhere. In addition, a new set of sufficient conditions for a stability comparison theorem have been formulated which to the best of the author's knowledge have not appeared elsewhere. Though the treatment in Section VI is somewhat brief, it is intended for the most part to complement the rest of the paper by illustrating some of the more useful properties exhibited by uniformly and restrictively stable systems. The theory of stability for non-linear non-autonomous systems is still quite incomplete and is today an area of extensive active research.

Some other aspects of stability theory not considered here are the Poincare-Bendixson theory for autonomous systems, the second method of Lyapunov, and various analytical approaches such as the method of Lindstedt, Krylov and Bogolyubov, and of Van Der Pol. As mentioned earlier, the approach to the problem of stability of a system of differential equations varies with both the form and order of the system. This paper has considered but one of these approaches, one which is particularly useful in the treatment of linear systems. For a treatment of some of the other areas in the theory of stability of ordinary differential equations, we refer the reader to several of the excellent texts on stability, including [2], [5] and [7].

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13. ABSTRACT

Stability in the sense of Lyapunov is investigated for systems of ordinary differential equations. Various established results are discussed and a new set of sufficient conditions for a stability comparison theorem is obtained. The treatment is principally concerned with linear systems, though some results for non-linear systems are considered briefly.

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